Chapter 2

The Equation $\dot{x} = Ax$

The object of study in this chapter is the unforced state equation

$$\dot{x} = Ax.$$

Here A is an $n \times n$ real matrix and $x(t)$ an n-dimensional vector-valued function of time.

2.1 Brief Review of Some Linear Algebra

In this brief section we review these concepts/results: $\mathbb{R}^n$, linear independence of a set of vectors, span of a set of vectors, subspace, basis for a subspace, rank of a matrix, existence and uniqueness of a solution to $Ax = b$ where A is not necessarily square, inverse of a matrix, invertibility. If you remember them (and I hope you do), skip to the next section.

The symbol $\mathbb{R}^n$ stands for the vector space of n-tuples, i.e., ordered lists of n real numbers.

A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is **linearly independent** if none is a linear combination of the others. One way to check this is to write the equation

$$c_1 v_1 + \cdots + c_k v_k = 0$$

and then try to solve for the $c_i's$. The set is linearly independent iff the only solution is $c_i = 0$ for every i,

The **span** of $\{v_1, \ldots, v_k\}$, denoted $\text{Span}\{v_1, \ldots, v_k\}$, is the set of all linear combinations of these vectors.

A **subspace** $\mathcal{V}$ of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ that is also a vector space in its own right. This is true iff these two conditions hold: If x, y are in $\mathcal{V}$, then so is $x + y$; if x is in $\mathcal{V}$ and c is a scalar, then cx is in $\mathcal{V}$. Thus $\mathcal{V}$ is closed under the operations of addition and scalar multiplication. In $\mathbb{R}^3$ the subspaces are the lines through the origin, the planes through the origin, the whole of $\mathbb{R}^3$, and the set consisting of only the zero vector.

A **basis** for a subspace is a set of linearly independent vectors whose span equals the subspace. The number of elements in a basis is the **dimension** of the subspace.

The **rank** of a matrix is the dimension of the span of its columns. This can be proved to equal the dimension of the span of its rows.

The equation $Ax = b$ has a solution iff b belongs to the span of the columns of A, equivalently

$$\text{rank } A = \text{rank} \begin{bmatrix} A & b \end{bmatrix}.$$

When a solution exists, it is unique iff the columns of A are linearly independent, that is, the rank of A equals its number of columns.

The **inverse** of a square matrix A is a matrix B such that $BA = I$. If this is true, then $AB = I$. The inverse is unique and we write A^{-1}. A square matrix A is invertible iff its rank equals its dimension (we say "A has full rank"); equivalently, its determinant is nonzero. The inverse equals the adjoint divided by the determinant.

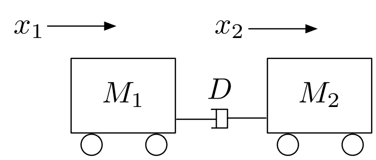
2.2 Eigenvalues and Eigenvectors

Now we turn to $\dot{x} = Ax$. The time evolution of $x(t)$ can be understood from the eigenvalues and eigenvectors of A—a beautiful connection between dynamics and algebra. Recall that the eigenvalue equation is

$$Av = \lambda v.$$

Here λ is a real or complex number and v is a nonzero real or complex vector; λ is an eigenvalue and v a corresponding eigenvector. The eigenvalues of A are unique but the eigenvectors are not: If v is an eigenvector, so is cv for any real number $c \neq 0$. The **spectrum** of A, denoted $\sigma(A)$, is its set of eigenvalues. The spectrum consists of n numbers, in general complex, and they are equal to the zeros of the characteristic polynomial $\det(sI - A)$.

Example Consider two carts and a dashpot like this:



Take $D = 1$, $M_1 = 1$, $M_2 = 1/2$, $x_3 = \dot{x}_1$, $x_4 = \dot{x}_2$. You can derive that the model is $\dot{x} = Ax$, where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}.$$

The characteristic polynomial of A is $s^3(s+3)$, and therefore

$$\sigma(A) = \{0, 0, 0, -3\}.$$

□

The equation $Av = \lambda v$ says that the action of A on an eigenvector is very simple—just multiplication by the eigenvalue. Likewise, the motion of $x(t)$ starting at an eigenvector is very simple.

Lemma 2.2.1 *If $x(0)$ is an eigenvector v of A and λ the corresponding eigenvalue, then $x(t) = e^{\lambda t} v$. Thus $x(t)$ is an eigenvector too for every t.*

Proof The initial-value problem

$$\dot{x} = Ax, \quad x(0) = v$$

has a unique solution—this is from differential equation theory. So all we have to do is show that $e^{\lambda t}v$ satisfies both the initial condition and the differential equation, for then $e^{\lambda t}v$ must be the solution $x(t)$. The initial condition is easy:

$$e^{\lambda t}v\Big|_{t=0} = v.$$

And for the differential equation,

$$\frac{d}{dt}(e^{\lambda t}v) = e^{\lambda t}\lambda v = e^{\lambda t}Av = A(e^{\lambda t}v).$$

□

The result of the lemma extends to more than one eigenvalue. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A and let $v_1, \dots, v_n$ be corresponding eigenvectors. Suppose the initial state $x(0)$ can be written as a linear combination of the eigenvectors:

$$x(0) = c_1 v_1 + \cdots + c_n v_n.$$

This is certainly possible for every $x(0)$ if the eigenvectors are linearly independent. Then the solution satisfies

$$x(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n.$$

This is called a **modal expansion** of $x(t)$.

Example

$$A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}, \quad \lambda_1 = 0, \ \lambda_2 = -3, \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Let's say $x(0) = (0, 1)$. The equation

$$x(0) = c_1 v_1 + c_2 v_2$$

is equivalent to

$$x(0) = Vc,$$

where V is the 2×2 matrix with columns v_1, v_2 and c is the vector (c_1, c_2). Solving gives $c_1 = c_2 = 1/3$. So

$$x(t) = \frac{1}{3}v_1 + \frac{1}{3}e^{-3t}v_2$$

□

The case of complex eigenvalues is only a little complicated. If λ_1 is a complex eigenvalue, some other, say λ_2, is its complex conjugate: $\lambda_2 = \overline{\lambda_1}$. The two eigenvectors, v_1 and v_2, can be taken to be complex conjugates too (easy proof). Then if $x(0)$ is real and we solve

$$x(0) = c_1 v_1 + c_2 v_2,$$

we'll find that c_1, c_2 are complex conjugates as well. Thus the equation will look like

$$x(0) = c_1 v_1 + \overline{c_1 v_2} = 2\Re\,(c_1 v_1),$$

where $\Re$ denotes real part.

Example

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \lambda_1 = j,\ \lambda_2 = -j, \quad v_1 = \begin{bmatrix} 1 \\ -j \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ j \end{bmatrix}$$

Suppose $x(0) = (0, 1)$. Then $c_1 = j/2$, $c_2 = -j/2$ and

$$x(t) = 2\Re\,\left(c_1 e^{\lambda_1 t} v_1\right) = \Re\,\left(j e^{jt} \begin{bmatrix} 1 \\ -j \end{bmatrix}\right) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}.$$

□

2.3 The Jordan Form

Now we turn to the structure theory of a matrix related to its eigenvalues. It's convenient to introduce a term, the **kernel** of a matrix A. Kernel is another name for nullspace. Thus Ker A is the set of all vectors x such that $Ax = 0$; that is, Ker A is the solution space of the homogeneous equation $Ax = 0$. Notice that the zero vector is always in the kernel. If A is square, then Ker A is the zero subspace, and we write Ker $A = 0$, iff 0 is not an eigenvalue of A. If 0 *is* an eigenvalue, then Ker A equals the span of all the eigenvectors corresponding to this eigenvalue; we say Ker A is the **eigenspace** corresponding to the eigenvalue 0. More generally, if λ is an eigenvalue of A the corresponding eigenspace is the solution space of $Av = \lambda v$, that is, of $(A - \lambda I)v = 0$, that is, Ker $(A - \lambda I)$.

Let's begin with the simplest case, where A is 2×2 and has 2 distinct eigenvalues, λ_1, λ_2. You can show (this is a good exercise) that there are then 2 linearly independent eigenvectors, say v_1, v_2 (maybe complex vectors). The equations

$$Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_2 v_2$$

are equivalent to the matrix equation

$$A \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

that is, $AV = VA_{JF}$, where

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}, \quad A_{JF} = \text{diag}\,(\lambda_1, \lambda_2).$$

The latter matrix is the **Jordan form** of A. It is unique up to reordering of the eigenvalues. The mapping $A \longmapsto A_{JF} = V^{-1}AV$ is called a similarity transformation. Example:

$$A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \quad A_{JF} = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}.$$

Corresponding to the eigenvalue $\lambda_1 = 0$ is the eigenvector $v_1 = (1,1)$, the first column of V. All other eigenvectors corresponding to λ_1 have the form cv_1, $c \neq 0$. We call the subspace spanned by v_1 the eigenspace corresponding to λ_1. Likewise, $\lambda_2 = -3$ has a one-dimensional eigenspace.

These results extend from $n = 2$ to general n. Note that in the preceding result we didn't actually need distinctness of the eigenvalues — only linear independence of the eigenvectors.

Theorem 2.3.1 *The Jordan form of A is diagonal, i.e., A is diagonalizable by similarity transformation, iff A has n linearly independent eigenvectors. A sufficient condition is n distinct eigenvalues.*

The great thing about diagonalization is that the equation $\dot{x} = Ax$ can be transformed via $w = V^{-1}x$ into $\dot{w} = A_{JF}w$, that is, n **decoupled** equations:

$$\dot{w}_i = \lambda_i w_i, \quad i = 1, \ldots, n.$$

The latter equations are trivial to solve:

$$w_i(t) = \mathrm{e}^{\lambda_i t} w_i(0), \quad i = 1, \ldots, n.$$

Now we look at how to construct the Jordan form when there are not n linearly independent eigenvectors. We start where A has only 0 as an eigenvalue.

Nilpotent matrices

Consider

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.1}$$

For both of these matrices, $\sigma(A) = \{0, 0, 0\}$. For the first matrix, the eigenspace Ker A is two-dimensional and for the second matrix, one-dimensional. These are examples of nilpotent matrices: A is **nilpotent** if $A^k = 0$ for some $k \geq 1$. The following statements are equivalent:

1. A is nilpotent.

2. All its eigs are 0.

3. Its characteristic polynomial is s^n.

4. It is similar to a matrix of the form (2.1), where all elements are 0's, except 0's or 1's on the first diagonal above the main one. This is called the Jordan form of the nilpotent matrix.

Example Suppose A is 3×3 and $A = 0$. Then of course it's already in Jordan form,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

□

Example Here we do an example of transforming a nilpotent matrix to Jordan form. Take

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}.$$

The rank of A is 3 and hence the kernel has dimension 2. We can compute that

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^4 = 0.$$

Take any vector v_5 in Ker $A^4 = \mathbb{R}^5$ that is not in Ker A^3, for example,

$$v_5 = (0, 0, 0, 0, 1).$$

Then take

$$v_4 = Av_5, \quad v_3 = Av_4, \quad v_2 = Av_3.$$

We get

$$v_4 = (0, 0, 0, 1, -1) \in \text{Ker } A^3, \quad \notin \text{Ker } A^4$$

$$v_3 = (0, 1, 0, 0, 0) \in \text{Ker } A^2, \quad \notin \text{Ker } A^3$$

$$v_2 = (1, -1, 0, 0, 0) \in \text{Ker } A, \quad \notin \text{Ker } A^2.$$

Finally, take $v_1 \in$ Ker A, linearly independent of v_2, for example,

$$v_1 = (0, 0, 1, 0, 0).$$

Assemble $v_1, \ldots, v_5$ into the columns of V. Then

$$V^{-1}AV = A_{JF} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is block diagonal, like this:

$$\left[\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$\square$

In general, the Jordan form of a nilpotent matrix has 0 in each entry except possibly in the first diagonal above the main diagonal which may have some 1s.

A nilpotent matrix has only the eigenvalue 0. Now consider a matrix A that has only one eigenvalue, λ, i.e.,

$$\det(sI - A) = (s - \lambda)^n.$$

To simplify notation, suppose $n = 3$. Letting $r = s - \lambda$, we have

$$\det[rI - (A - \lambda I)] = r^3,$$

i.e., $A - \lambda I$ has only the zero eigenvalue, and hence $A - \lambda I =: N$, a nilpotent matrix. So the Jordan form of N must look like

$$\begin{bmatrix} 0 & \star & 0 \\ 0 & 0 & \star \\ 0 & 0 & 0 \end{bmatrix},$$

where each star can be 0 or 1, and hence the Jordan form of A is

$$\begin{bmatrix} \lambda & \star & 0 \\ 0 & \lambda & \star \\ 0 & 0 & \lambda \end{bmatrix}, \tag{2.2}$$

To recap, if A has just one eigenvalue, λ, then its Jordan form is $\lambda I + N$, where N is a nilpotent matrix in Jordan form.

An extension of this analysis results in the **Jordan form** in general. Suppose A is $n \times n$ and $\lambda_1, \ldots, \lambda_p$ are the distinct eigenvalues of A and $m_1, \ldots, m_p$ are their multiplicities; that is, the characteristic polynomial is

$$\det(sI - A) = (s - \lambda_1)^{m_1} \cdots (s - \lambda_p)^{m_p}.$$

Then A is similar to

$$A_{JF} = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{bmatrix},$$

where A_i is $m_i \times m_i$ and it has only the eigenvalue λ_i. Thus A_i has the form $\lambda_i I + N_i$, where N_i is a nilpotent matrix in Jordan form. Example:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

As we saw, the spectrum is $\sigma(A) = \{0, 0, 0, -3\}$. Thus the Jordan form must be of the form

$$A_{JF} = \begin{bmatrix} 0 & \star & 0 & 0 \\ 0 & 0 & \star & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

Since A has rank 2, so does A_{JF}. Thus only one of the stars is 1. Either is possible, for example,

$$A_{JF} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

This has two "Jordan blocks":

$$A_{JF} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = -3.$$

2.4 The Transition Matrix

Let us review from the ECE356 course notes. For a square matrix M, the exponential e^M is defined as

$$e^M := I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots.$$

The matrix e^M is not the same as the component-wise exponential of M. Facts:

1. e^M is invertible for every M, and $(e^M)^{-1} = e^{-M}$.
2. $e^{M+N} = e^M e^N$ iff M and N commute, i.e., $MN = NM$.

The matrix function $t \longmapsto e^{tA} : \mathbb{R} \to \mathbb{R}^{n \times n}$ is then defined and is called the **transition matrix** associated with A. It has the properties

1. $e^{tA}|_{t=0} = I$
2. e^{tA} and A commute.
3. $\dfrac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A$.

Moreover, the solution of

$$\dot{x} = Ax, \quad x(0) = x_0$$

is $x(t) = e^{tA}x_0$. So e^{tA} maps the state at time 0 to the state at time t. In fact, it maps the state at any time t_0 to the state at time $t_0 + t$.

On computing the transition matrix

via the Jordan form If one can compute the Jordan form of A, then e^{tA} can be written in closed form, as follows. The equation

$$AV = VA_{JF}$$

implies

$$A^2V = AVA_{JF} = VA_{JF}^2.$$

Continuing in this way gives

$$A^kV = VA_{JF}^k,$$

and then

$$e^{At}V = Ve^{A_{JF}t},$$

so finally

$$e^{At} = Ve^{A_{JF}t}V^{-1}.$$

The matrix exponential $e^{A_{JF}t}$ is easy to write down. For example, suppose there's just one eigenvalue, so $A_{JF} = \lambda I + N$, N nilpotent, $n \times n$. Then

$$\begin{aligned} e^{A_{JF}t} &= e^{\lambda t}e^{Nt} \\ &= e^{\lambda t}\left(I + Nt + N^2\frac{t^2}{2!} + \cdots + N^{n-1}\frac{t^{n-1}}{(n-1)!}\right). \end{aligned}$$

via Laplace transforms Taking Laplace transforms of

$$\dot{x} = Ax, \quad x(0) = x_0$$

gives

$$sX(s) - x_0 = AX(s).$$

This yields

$$X(s) = (sI - A)^{-1}x_0.$$

Comparing

$$x(t) = e^{tA}x_0, \quad X(s) = (sI - A)^{-1}x_0$$

shows that e^{tA}, $(sI - A)^{-1}$ are Laplace transform pairs. So one can get e^{tA} by finding the matrix $(sI - A)^{-1}$ and then taking the inverse Laplace transform of each element.

2.5 Stability

The concept of stability is fundamental in control engineering. Here we look at the scenario where the system has no input, but its state has been perturbed and we want to know if the system will recover. This was introduced in the ECE356 course notes. Here we go a little farther now that we're armed with the Jordan form.

The maglev example is a good one to illustrate this point. Suppose a feedback controller has been designed to balance the ball's position at 1 cm below the magnet. Suppose if the ball is placed at **precisely** 1 cm it will stay there; that is, the 1 cm location is a closed-loop equilibrium point. Finally, suppose there is a temporary wind gust that moves the ball away from the 1 cm position. The stability questions are, will the ball move back to the 1 cm location; if not, will it at least stay near that location?

So consider

$$\dot{x} = Ax.$$

Obviously if $x(0) = 0$, then $x(t) = 0$ for all t. We say the origin is an **equilibrium point**—if you start there, you stay there. Equilibrium points can be stable or not. While there are more elaborate and formal definitions of stability for the above homogeneous system, we choose the following two: The origin is **asymptotically stable** if $x(t) \longrightarrow 0$ as $t \longrightarrow \infty$ for all $x(0)$. The origin is **stable** if $x(t)$ remains bounded as $t \longrightarrow \infty$ for all $x(0)$. Since $x(t) = \mathrm{e}^{At}x(0)$, the origin is asymptotically stable iff every element of the matrix e^{At} converges to zero, and is stable iff every element of the matrix e^{At} remains bounded as $t \longrightarrow \infty$. Of course, asymptotic stability implies stability.

Asymptotic stability is relatively easy to characterize. Using the Jordan form, one can prove this very important result, where $\Re$ denotes "real part":

Theorem 2.5.1 *The origin is asymptotically stable iff the eigenvalues of A all satisfy $\Re\, \lambda < 0$.*

Let's say the matrix A is **stable** if its eigenvalues satisfy $\Re\, \lambda < 0$. Then the origin is asymptotically stable iff A is stable.

Now we turn to the more subtle property of stability. We'll do some examples, and we may as well have A in Jordan form.

Consider the nilpotent matrix

$$A = N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Obviously, $x(t) = x(0)$ for all t and so the origin is stable. By contrast, consider

$$A = N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\mathrm{e}^{Nt} = I + tN,$$

which is unbounded and so the origin is not stable. This example extends to the $n \times n$ case: If A is nilpotent, the origin is stable iff $A = 0$.

Here's the test for stability in general in terms of the Jordan form of A:

$$A_{JF} = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{bmatrix}.$$

Recall that each A_i has just one eigenvalue, λ_i, and that $A_i = \lambda_i I + N_i$, where N_i is a nilpotent matrix in Jordan form.

Theorem 2.5.2 *The origin is stable iff the eigenvalues of A all satisfy $\Re\ \lambda \leq 0$ and for any eigenvalue with $\Re\ \lambda_i = 0$, the nilpotent matrix N_i is zero, i.e., A_i is diagonal.*

Here's an example with complex eigenvalues:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A_{JF} = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}.$$

The origin is stable since there are two 1×1 Jordan blocks. Now consider

$$A = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues are $j, j, -j, -j$ and so the Jordan form must look like

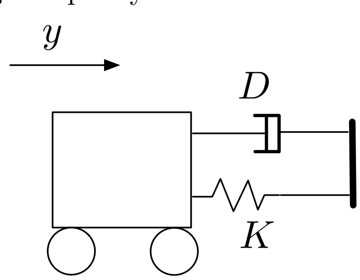
$$A_{JF} = \begin{bmatrix} j & \star & 0 & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & -j & \star \\ 0 & 0 & 0 & -j \end{bmatrix}.$$

Since the rank of $A - jI$ equals 3, the upper star is 1; since the rank of $A + jI$ equals 3, the lower star is 1. Thus

$$A_{JF} = \begin{bmatrix} j & 1 & 0 & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & -j & 1 \\ 0 & 0 & 0 & -j \end{bmatrix}.$$

Since the Jordan blocks are not diagonal, the origin is not stable.

Example Consider the cart-spring-damper system



The equation is

$$M\ddot{y} + D\dot{y} + Ky = 0.$$

Defining $x = (y, \dot{y})$, we have $\dot{x} = Ax$ with

$$A = \begin{bmatrix} 0 & 1 \\ -K/M & -D/M \end{bmatrix}.$$

Assume $M > 0$ and $K, D \geq 0$. If $D = K = 0$, the eigenvalues are $\{0, 0\}$ and A is a nilpotent matrix in Jordan form. The origin is an unstable equilibrium. If only $D = 0$ or $K = 0$ but not both, the origin is stable but not asymptotically stable. And if both D, K are nonzero, the origin is asymptotically stable. $\Box$

Example Two points move on the line $\mathbb{R}$. The positions of the points are x_1, x_2. They move toward each other according to the control laws

$$\dot{x}_1 = x_2 - x_1, \quad \dot{x}_2 = x_1 - x_2.$$

Thus the state is $x = (x_1, x_2)$ and the state equation is

$$\dot{x} = Ax, \quad A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = 0, \lambda_2 = -2$, so the origin is stable but not asymptotically stable. Obviously, the two points tend toward each other; that is, the state $x(t)$ tends toward the subspace

$$\mathcal{V} = \{x : x_1 = x_2\}.$$

This is the eigenspace for the zero eigenvalue. To see this convergence, write the initial condition as a linear combination of eigenvectors:

$$x(0) = c_1 v_1 + c_2 v_2, \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then

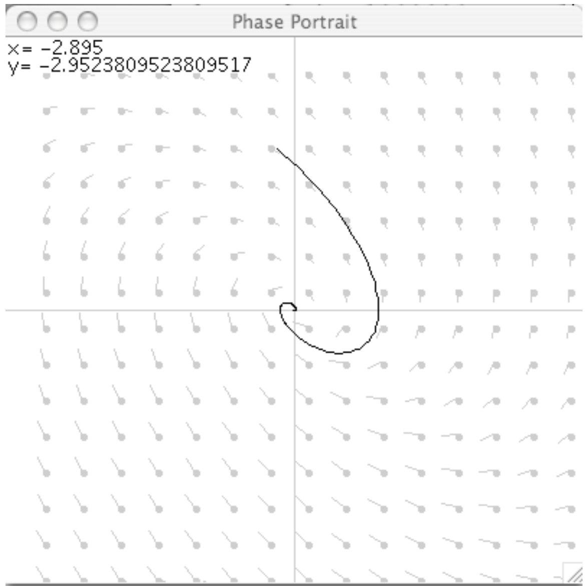
$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 = c_1 v_1 + c_2 e^{-2t} v_2 \to c_1 v_1.$$

So $x_1(t)$ and $x_2(t)$ both converge to c_1, the same point. $\Box$

Phase portraits help us visualize state evolution and stability, but they're applicable only for the $n = 2$ case. Below is shown a plot in $\mathbb{R}^2$ of the vector field for

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},$$

that is, at a grid of points, the directions of the velocity vectors Ax are shown translated to the point x. By following the arrows, we get a trajectory; one is shown. The plot was done using www.math.psu.edu/melvin/phase/newphase.html



You can also use MATLAB, Scilab (free), Mathematica, or Octave (free).

2.6 Problems

1. Are the following vectors linearly independent?

$$v_1 = (1, 1, 2, 0), \quad v_2 = (1, 0, 2, -2), \quad v_3 = (-1, 2, -2, 6).$$

2. Continuing with the same vectors, find a basis for Span $\{v_1, v_2, v_3\}$.

3. What kind of geometric object is $\{x : Ax = b\}$ when $A \in \mathbb{R}^{m \times n}$? That is, is it a sphere, a point—what?

4. (a) Let A be an 8×8 real matrix with eigenvalues

$$2, 2, -3, -3, -3, 8, 4, 4.$$

Assume

$$\operatorname{rank}(A - 2I) = 7, \ \operatorname{rank}(A + 3I) = 6, \ \operatorname{rank}(A - 4I) = 6.$$

Write down the Jordan form of A.

(b) The matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \end{bmatrix}$$

is nilpotent. Write down its Jordan form.

5. Take

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}.$$

Show that the matrix V constructed as follows satisfies $V^{-1}AV = A_{JF}$:

Select v_3 in Ker A^2 but not in Ker A.

Set $v_2 = Av_3$.

Select v_1 in Ker A such that $\{v_1, v_2\}$ is linearly independent.

Select an eigenvector v_4 corresponding to the eigenvalue -3.

Set $V = [v_1 \ \ v_2 \ \ v_3 \ \ v_4]$.

(The general construction of the basis for the Jordan form is along these lines.)

6. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 2 \end{bmatrix}.$$

Write down the Jordan form of A.

7. Consider

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix},$$

where σ and $\omega \neq 0$ are real. Find the Jordan form and the transition matrix.

8. In the previous problem, we saw that when

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

its transition matrix is easy to write down. This problem demonstrates that a matrix with distinct complex eigenvalues can be transformed into the above form using a nonsingular transformation. Let

$$A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}.$$

Determine the eigenvalues and eigenvectors of A, noting that they form complex conjugate pairs. Let the first eigenvalue be written as $a + jb$ with the corresponding eigenvector $v_1 + jv_2$. Take v_1 and v_2 as the columns of a matrix V. Find $V^{-1}AV$.

9. Consider the homogeneous state equation $\dot{x} = Ax$ with

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

and $x_0 = (3, 2)$. Find a modal expansion of $x(t)$.

10. Show that the origin is asymptotically stable for $\dot{x} = Ax$ iff all poles of every element of $(sI - A)^{-1}$ are in the open left half-plane. Show that the origin is stable iff all poles of every element of $(sI - A)^{-1}$ are in the closed left half-plane and those on the imaginary axis have multiplicity 1.

11. Consider the linear system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{aligned}$$

   (a) If $u(t)$ is the unit step and $x(0) = 0$, is $y(t)$ bounded?

   (b) If $u(t) = 0$ and $x(0)$ is arbitrary, is $y(t)$ bounded?

12. (a) Suppose that $\sigma(A) = \{-1, -3, -3, -1 + j2, -1 - j2\}$ and the rank of $(A - \lambda I)_{\lambda=-3}$ is 4. Determine A_{JF}.

   (b) Suppose that $\sigma(A) = \{-1, -2, -2, -2\}$ and the rank of $(A - \lambda I)_{\lambda=-2}$ is 3. Determine A_{JF}.

   (c) Suppose that $\sigma(A) = \{-1, -2, -2, -2, -3\}$ and the rank of $(A - \lambda I)_{\lambda=-2}$ is 3. Determine A_{JF}.

13. Find A_{JF} for

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}.$$

14. Summarize all the ways to find $\exp(At)$. Then find $\exp(At)$ for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

15. Consider the set

$$\{cv : c \geq 0\},$$

where $v \neq 0$ is a given vector in $\mathbb{R}^2$. This set is called a **ray** from the origin in the direction of v. More generally,

$$\{x_0 + cv : c \geq 0\}$$

is a ray from x_0 in the direction of v. Find a 2×2 matrix A and a vector x_0 such that the solution $x(t)$ of $\dot{x} = Ax,\ x(0) = x_0$ is a ray.

16. Consider the following system:

$$
\begin{aligned}
\dot{x}_1 &= -x_2 \\
\dot{x}_2 &= x_1 - 3x_2
\end{aligned}
$$

Do a phase portrait using Scilab or MATLAB. Interpret the phase portrait in terms of the modal decomposition of the system. Do lots more examples of this type.