ECE557 Systems Control

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Preface

This is the second Engineering Science course on control. It assumes ECE356 as a prerequisite. If you didn't take ECE356, you **must** go through Chapters 2 and 3 of the ECE356 course notes.

This course is on the state-space approach to control system analysis and design. By contrast, ECE356 treated frequency domain methods. Generally speaking, the state-space methods scale better to higher order, multi-input/output systems. The frequency domain methods use complex function theory; the state-space approach uses linear algebra—eigenvalues, subspaces, and all that.

The emphasis in the lectures will be on concepts, examples, and use of the theory.

There are several computer applications for solving numerical problems in this course. The most widely used is MATLAB, but it's expensive. I like Scilab, which is free. Others are Mathematica (expensive) and Octave (free).

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Chapter 1

Introduction

Control is that beautiful part of system science/engineering where we get to design part of the system, the controller, so that the system performs as intended. Control is a very rich subject, ranging from pure theory (Can a robot with just vision sensors be programmed to ride a unicycle?) down to the writing of real-time code. This course is mathematical, but that doesn't imply it is only theoretical and isn't applicable to real problems.

You are assumed to know Chapters 2 and 3 of the ECE356 course notes. This chapter gives a brief review of only part and is not sufficient.

First, some notation: Usually, a vector is written as a column vector, but sometimes to save space it is written as an n-tuple:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ or } x = (x_1, \ldots, x_n).$$

1.1 State Models

Systems that are linear, time-invariant, causal, finite-dimensional, and having proper transfer functions have state models,

$$\dot{x} = Ax + Bu, \quad y = Cx + Du.$$

Here u, x, y are vector-valued functions of t and A, B, C, D real constant matrices.

Deriving State Models

How to get a state model depends on what we have to start with.

Example n^{th} order ODE. Suppose we have the system

$$2\ddot{y} - \dot{y} + 3y = u.$$

The natural state vector is

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} =: \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \frac{1}{2}x_2 - \frac{3}{2}x_1 + \frac{1}{2}u\ ,\end{aligned}$$

so

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix},\quad B = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix},\quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},\quad D = 0.$$

This technique extends to

$$a_n y^{(n)} + \cdots + a_1\dot{y} + a_0 y = u.$$

What about derivatives on the right-hand side:

$$2\ddot{y} - \dot{y} + 3y = \dot{u} - 2u?$$

The transfer function is

$$Y(s) = \frac{s-2}{2s^2 - s + 3}U(s).$$

Introduce an intermediate signal v:

$$Y(s) = (s-2)\underbrace{\frac{1}{2s^2 - s + 3}U(s)}_{=:V(s)}.$$

Then

$$\begin{aligned}&2\ddot{v} - \dot{v} + 3v = u\\ &y = \dot{v} - 2v.\end{aligned}$$

Taking $x = (v, \dot{v})$ we get

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix},\quad B = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix},\quad C = \begin{bmatrix} -2 & 1 \end{bmatrix},\quad D = 0.$$

This technique extends to

$$y^{(n)} + \cdots + a_1\dot{y} + a_0 y = b_{n-1}u^{(n-1)} + \cdots + b_0 u.$$

The transfer function is

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}.$$

Then

$$G(s) = C(sI - A)^{-1}B,$$

where

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & \cdots & b_{n-1} \end{bmatrix}.$$

This state model is called the **controllable (canonical) realization** of $G(s)$.

For the case $n = m$, you divide denominator into numerator and thereby factor $G(s)$ into the sum of a constant and a strictly proper transfer function. This gives $D \neq 0$, namely, the constant.

If $m > n$, there is no state model.

What if we have two inputs u_1, u_2, two outputs y_1, y_2, and coupled equations such as

$$\ddot{y}_1 - \dot{y}_1 + \dot{y}_2 + 3y_1 = u_1 + u_2$$

$$2\frac{d^3y_2}{dt^3} - \dot{y}_1 + \dot{y}_2 + 4y_1 = u_2?$$

The natural state is

$$x = (y_1, \dot{y}_1, y_2, \dot{y}_2, \ddot{y}_2).$$

Please complete this example. □

Let's study the transfer matrix for the state model

$$\dot{x} = Ax + Bu, \quad y = Cx + Du.$$

Take Laplace transforms with zero initial conditions:

$$sX(s) = AX(s) + BU(s), \quad Y(s) = CX(s) + DU(s).$$

Eliminate $X(s)$:

$$(sI - A)X(s) = BU(s)$$

$$\Rightarrow X(s) = (sI - A)^{-1}BU(s)$$

$$\Rightarrow Y(s) = \underbrace{[C(sI - A)^{-1}B + D]}_{\text{transfer matrix}} U(s).$$

This leads to the **realization problem**: Given $G(s)$, find A, B, C, D such that

$$G(s) = C(sI - A)^{-1}B + D.$$

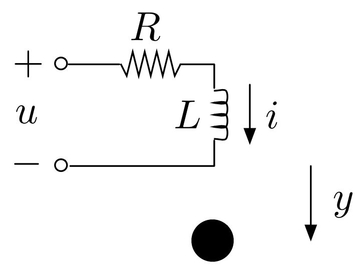
A solution exists iff $G(s)$ is rational and proper (every element of $G(s)$ has deg denom $\geq$ deg num). The solution is never unique.

There are general procedures for getting a state model, but we choose not to cover this topic in the interests of moving to other interests.

1.2 Examples

Here we look at two examples that we'll use repeatedly for illustration.

1.2.1 Magnetic levitation



This example was used frequently in ECE356. Imagine an electromagnet suspending an iron ball. Let the input be the voltage u and the output the position y of the ball below the magnet; let i denote the current in the circuit. Then

$$L\frac{di}{dt} + Ri = u.$$

Also, it can be derived that the magnetic force on the ball has the form Ki^2/y^2, K a constant. Thus

$$M\ddot{y} = Mg - K\frac{i^2}{y^2}.$$

Realistic numerical values are $M = 0.1$ Kg, $R = 15$ ohms, $L = 0.5$ H, $K = 0.0001$ Nm^2/A^2, $g = 9.8$ m/s^2. Substituting in these numbers gives the equations

$$0.5\frac{di}{dt} + 15i = u$$

$$0.1\frac{d^2y}{dt^2} = 0.98 - 0.0001\frac{i^2}{y^2}.$$

Define state variables $x = (x_1, x_2, x_3) = (i, y, \dot{y})$. Then the nonlinear state model is $\dot{x} = f(x, u)$, where

$$f(x, u) = (-30x_1 + 2u, x_3, 9.8 - 0.001x_1^2/x_2^2).$$

Suppose we want to stabilize the ball at $y = 1$ cm, or 0.01 m. We need a linear model valid in the neighbourhood of that value. Solve for the equilibrium point $(\bar{x}, \bar{u})$ where $\bar{x}_2 = 0.01$:

$$-30\bar{x}_1 + 2\bar{u} = 0, \quad \bar{x}_3 = 0, \quad 9.8 - 0.001\bar{x}_1^2/0.01^2 = 0.$$

Thus

$$\bar{x} = (0.99, 0.01, 0), \quad \bar{u} = 14.85.$$

The linearized model is

$$\dot{\delta x} = A\delta x + B\delta u, \quad \delta y = C\delta x,$$

where A equals the Jacobian of f with respect to x, evaluated at $(\bar{x}, \bar{u})$, and B equals the same except with respect to u:

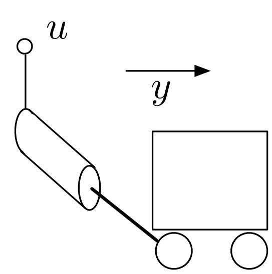
$$A = \begin{bmatrix} -30 & 0 & 0 \\ 0 & 0 & 1 \\ -0.002x_1/x_2^2 & 0.002x_1^2/x_2^3 & 0 \end{bmatrix}_{(\bar{x},\bar{u})} = \begin{bmatrix} -30 & 0 & 0 \\ 0 & 0 & 1 \\ -19.8 & 1940 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of A are $-30, \pm 44.05$, the units being s^{-1}. The corresponding time constants are $1/30 = 0.033, 1/44.05 = 0.023$ s. The first is the time constant of the electric circuit; the second, the time constant of the magnetics.

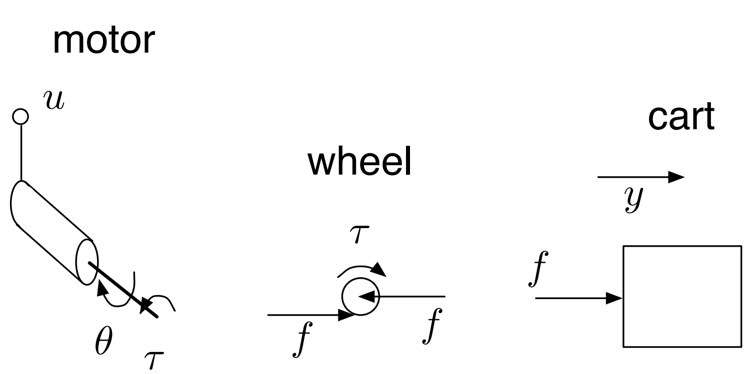
1.2.2 Vehicles

The second example is a vehicle control problem motivated by research on intelligent highway systems. We begin with the simplest vehicle, a cart with a motor driving one wheel:



The input is the voltage u to the motor, the output the cart position y. We want the model from u to y.

Free body diagrams:



The cart

A force f via the wheel through the axle:

$$M\ddot{y} = f. \tag{1.1}$$

The wheel

An equal and opposite force f at the axle; a horizontal force where the wheel contacts the floor. If the inertia of the wheel is negligible, the two horizontal forces are equal. Finally, a torque τ from the motor. Equating moments about the axle gives $\tau = fr$, where r is the radius of the wheel. Thus

$$f = \tau / r. \tag{1.2}$$

The motor

The electric circuit equation is

$$L\frac{di}{dt} + Ri = u - v_b, \tag{1.3}$$

where v_b is the back emf. The torque produced by the motor:

$$\tau_m = Ki. \tag{1.4}$$

Newton's second law for the motor shaft:

$$J\ddot{\theta} = \tau_m - \tau. \tag{1.5}$$

The back emf is

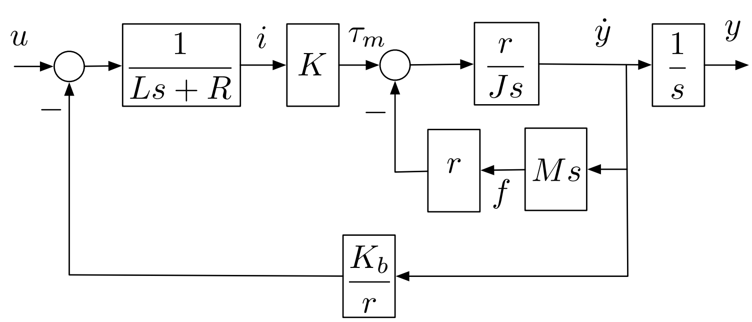
$$v_b = K_b\dot{\theta}. \tag{1.6}$$

Finally, the relationship between shaft angle and cart position:

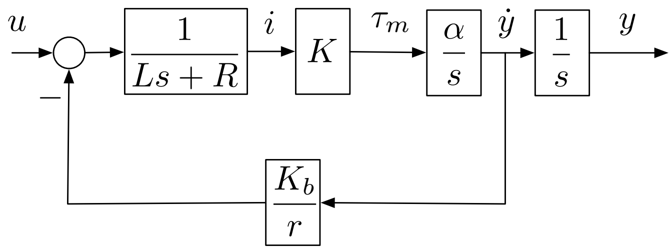
$$y = r\theta. \tag{1.7}$$

Combining

The block diagram is then

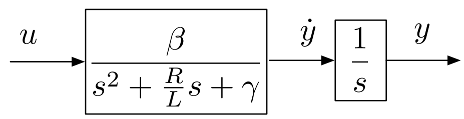


The inner loop can be reduced, giving



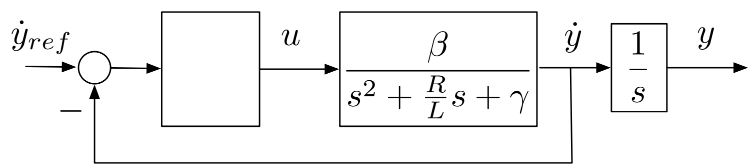
$$\alpha = \frac{r}{r^2 M + J}$$

Finally, we have the third-order system



$$\beta = \frac{\alpha K}{L} \qquad \gamma = \frac{\alpha K K_b}{rL}$$

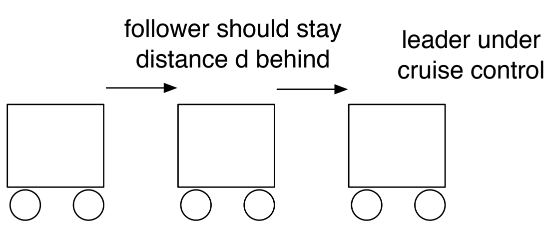
Although this vehicle is very easy to control, for more complex vehicles (Jeeps on open terrain) it's customary to design a loop to cancel the dynamics, leaving a simpler kinematic vehicle, like this:



If the loop is well designed, that is, $\dot{y}_{ref} \approx \dot{y}$, we can regard the system as merely a kinematic point, with input, velocity, say v, and output, position, y.

Platoons

Now suppose there are several of these motorized carts. We want them to move in a straight line like this: A designated leader should go at constant speed: the second should follow at a fixed distance d; the third should follow the second at the distance d; and so on.



We'll return to this problem later.

1.3 Problems

The first few problems study the concept of linearity of a system. Recall that a system $\mathbf{F}$ with input u and output y is **linear** if it satisfies two conditions: superposition, i.e.,

$$\mathbf{F}(u_1 + u_2) = \mathbf{F}(u_1) + \mathbf{F}(u_2),$$

and homogeneity,

$$\mathbf{F}(cu) = c\mathbf{F}(u),$$

c a real constant. To prove it's not linear, you have to give a counterexample for one of these two conditions.

1. Consider a quantizer $\mathbf{Q}$ with input $u(t)$, that can take on a continuum of values, and output $y(t)$, which can take on only countably many values, say, $\{b_k\}_{k\in\mathbb{Z}}$. More specifically, suppose $\mathbb{R}$ is partitioned into intervals I_k, $k \in \mathbb{Z}$, and if $u(t) \in I_k$, then $y(t) = b_k$. Prove that $\mathbf{Q}$ is not linear.

2. Let $\mathbf{S}$ denote the ideal sampler of sampling period T; it maps a continuous-time signal $u(t)$ into the discrete-time signal $u[k] = u(kT)$. Let $\mathbf{H}$ denote the synchronized zero-order hold; it maps a discrete-time signal $y[k]$ into $y(t)$, where

   $$y(t) = y[k], \quad kT \le t < (k+1)T.$$

   Then $\mathbf{HS}$ maps $u(t)$ to $y(t)$ where

   $$y(t) = u(kT), \quad kT \le t < (k+1)T.$$

   Is $\mathbf{HS}$ linear? If so, prove it; if not, give a counterexample.

3. Consider the amplitude modulation system with input $u(t)$ and output $y(t) = u(t)\cos(t)$. Is it linear?

4. At time $t = 0$ a force $v(t)$ is applied to a mass M whose position is $y(t)$; the mass is initially at rest. Thus $M\ddot{y} = v$, where $y(0) = \dot{y}(0) = 0$. The force is the output of a saturating actuator with input $u(t)$ in this way:

   $$v = \begin{cases} u, & -1 \le u \le 1 \\ 1, & u > 1 \\ -1, & u < -1. \end{cases}$$

   Is the system from u to y linear?

5. Give an example of a system that is linear, infinite-dimensional, causal, and time-varying.

6. Express the superposition property of a system $\mathbf{F}$ in terms of a block diagram. Express the homogeneity property in like manner.

7. Both by hand and by Scilab/MATLAB find a state model for the system with transfer function

   $$G(s) = \frac{s^3 - 1}{2s^3 + s^2 - 2s}.$$

8. Consider the system model $\dot{x} = Ax + Bu$, $y = Cx$ with

$$A = \begin{bmatrix} 1 & 2.5 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & -3 & 1 & 0 \\ 0 & -0.5 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix}.$$

Both by hand and by Scilab/MATLAB find the transfer function from u to y.

9. Kirchoff's laws for a circuit lead to algebraic constraints (e.g., currents into a node sum to zero). Consider a system with inputs u_1, u_2 and outputs y_1, y_2 governed by the equations

$$\ddot{y}_1 + 2\dot{y}_1 + y_2 = u_1$$

$$y_1 + y_2 = u_2.$$

Find the transfer matrix from $u = (u_1, u_2)$ to $y = (y_1, y_2)$. Does this system have a state model? If so, find one.

10. Consider the system with input $u(t)$ and output $y(t)$ where

$$4\ddot{y} + \dot{y}^2 - y = (3t^2 + 8)u.$$

The nominal input and output are $u_0(t) = 1$, $y_0(t) = t^2$ (you can check that they satisfy the differential equation). Derive a nonlinear state model of the form

$$\dot{x} = f(x, u, t).$$

Linearize this about the nominal state and input, ending up with a linear state equation.

11. An unforced pendulum is modeled by the equation

$$L\,\ddot{\theta} + g \sin\theta = 0,$$

where L = length, g = gravity constant, θ = angle of pendulum.

   (a) Put this model in the form of a state equation.
   (b) Find all equilibrium points.
   (c) Find the linearized model for each equilibrium point.

12. A system has three inputs u_1, u_2, and u_3 and three outputs y_1, y_2, and y_3. The equations are

$$\begin{aligned} \dddot{y}_1 + a_1\ddot{y}_1 + a_2(\dot{y}_1 + \dot{y}_2) + a_3(y_1 - y_3) &= u_1 \\ \ddot{y}_2 + a_4(\dot{y}_2 - \dot{y}_1 + 2\dot{y}_3) + a_5(y_2 - y_1) &= u_2 \\ \dot{y}_3 + a_6(y_3 - y_1) &= u_3\,. \end{aligned}$$

Find a state-space model for this system.

13. Find two different state models for the system

$$\ddot{y} + a\dot{y} + by = u + c\dot{u}.$$